A Celebration of Kepler’s Third Law: Harmony, Power Laws, and Keplerian Orbits

J. McKim Malville

Abstract. Kepler’s greatest achievements are contained in his three laws, which today reappear constantly at the cutting edge of modern astrophysics. The modified third law has been the major workhorse of astronomy and astrophysics in determining the mass of objects throughout the universe. Kepler never revealed why he chose the integers, 2 and 3, to test using Tycho’s data. Because of his attachment to Pythagorean ideas and his knowledge of musical theory, I suggest he was guided by the musical interval known as the perfect fifth, for which the ratio of frequencies is 3/2. The perfect fifth is the most consonant of all intervals except the octave, and, as such, is the basis of all the tuning of stringed instruments. If Kepler had recognized the significance of the perfect fifth in analyzing Tycho’s data, it suggests a very pleasing historical parallelism between music and astronomy. In discovering the third law, Kepler also chanced upon the world's first known power law, which is now found in many forms throughout the earth and heavens. In discovering that the Galilean moons of Jupiter also obeyed the third law, Kepler encountered the phenomenon of scale independence, which is responsible for the ubiquity of power laws across the universe. The third law also played a crucial role in Newton’s discovery of the inverse square law of gravity in 1666. Not only did it provide Newton with a crucial mathematical step, but the third law also had the authority of Tyco’s observations. Christopher Wren and Edmund Halley relied similarly upon the third law for their apparent independent discoveries of the inverse square law.

Introduction

The greatest, long-lasting contributions of Johannes Kepler in astronomy are contained in his three laws of planetary motion.

1. Planets follow ellipses with the sun at one locus.
2. Planets sweep out equal areas in equal time.
3. The period of a planet, P, is proportional to the length of the semi-major axis, a, raised to the 3/2 power of its orbit, or $P=ka^{3/2}$.

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When we teach these three laws in our introductory astronomy courses, the third law containing its 3/2 power is often the source cause of consternation and distress to our students. How did Kepler choose the ratio 3/2, they may ask and why are they integers if they are being matched with real data?

The third law was crucial in Newton's development of the Law of Gravity. Without it, Newton would have been unable to assert that his law had the legitimacy of verification by observational data.[1] Kepler based his form of the law upon the observational data of Tycho Brahe. Modified by Newton into $M = ka^3/P^2$, it has provided essential in determining the mass of practically everything astronomical from planets, stars, galaxies, supermassive black holes, and clusters of galaxies. Because of roles that Brahe and Kepler played in the formation of the law of gravity, it is clear that Newton stood on the shoulders of giants.[2] It must be the first data-driven power law in physics and astronomy of which there are countless examples in the natural world.[3]The vast sweep of the third law is demonstrated in determination of the mass of planets, stars, and, even, those supermassive black holes that generate gravity waves.[4] Belief in the Keplerian essence of planetary obits continued unabated until the anomalous precession of Mercury was revealed by Einstein to be due to the effects of General Relativity.[5] Now the obits of Mercury, and to a much lesser extent those of Venus and the Earth, can be viewed as a combination of Newtonian mechanics and General Relativity. The extreme influence of General Relativity on Keplerian orbits is found in the precession of stars orbiting the supermassive black hole in the center of our galaxy, which are precessing considerably faster than Mercury[6].

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[1] Richard Westfall, *Never at Rest* (Cambridge: Cambridge University Press,1980) p.152; Kenneth Ford, *Basic Physics* (Waltham: Blaisdell, 1968) pp.352–361.
[2] Isaac Newton, "Letter from Sir Isaac Newton to Robert Hooke". *Historical Society of Pennsylvania*. Retrieved 7 June 2018.
[3] James Sethna, 'Power Laws in Physics', *Nature Reviews* 4, pp.501–503, (2022); Per Bak, *How Nature Works* (New York: Copernicus, 1996)
[4] Priyamvada Natarajan, *Mapping the Universe: The Radical Scientific Ideas that Reveal the Cosmos* (New Haven, CT: Yale University Press, 2016) pp. 84-85, 92, 93.
[5] Thomas Levenson, *The Hunt for Vulcan* (New York: Random House, 2016)
[6] S. Gillessen, F. Eisenhauer, S. Trippe, T. Alexander, R. Genzel, F. Martins, and T. Ott, 'Monitoring Stellar Orbits Around The Massive Black Hole In The Galactic Center', *The Astrophysical Journal*, 692 (2009): pp.1075–110; A.M. Ghez, S. Salim, N.N. Weinberg, et al., 'Measuring Distance and Properties of the

The Perfect Fifth and Power Laws

The **third law** is a remarkably successful synthesis of mathematics, music, and astronomy. Kepler's first first job was in Gratz where he taught mathematics between 1594 and 1600. He was wise enough mathematician to recognized that there is not a linear relationship between the periods and distances of planets from the sun. His choice of testing a power law for that relation must be a case inductive reasoning on his part. Then, he needed to find what exponents might be consistent Tycho Brahe's data. How did he choose the exponents that he could test? Although he ended up with 3/2, which is the perfect fifth, I suggest that his final choice was not the result of a mystical epiphany or a sudden bright idea, but the result from his knowledge of musical theory and methodical testing of various musical intervals. With this emphasis on musical theory, I depart from interpretations that the 3/2 ratio came exclusively from mathematics and astronomy.[7] One may ask how much musical theory Kepler knew; was he really acquainted with the perfect fifth; and did the perfect fifth quench Kepler's thirst for understanding the music of the spheres?

In early childhood, starting as early as age five, Kepler was immersed in the musical traditions of Württemberg, practicing German psalmody as well as the Latin sequences and hymns. He must have become familiar with the perfect fifth as he learned about Pythagorean tuning of stringed instruments, as daily singing was accompanied with weekly lessons in theory.[8] Dickreiter concludes that Kepler derived a solid knowledge of music theory from his primary schooling, which 'deepened during his theological studies as an adult."[9] Later in life his knowledge of musical theory is clearly evident in *Harmonice mundi,* in which he attempts to understand the fundamental nature of musical harmony.[10] Kepler believed in the importance of building his hypothesis on observations.[11] In this case

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Milky Way's Central Supermassive Black Hole with Stellar Orbits', *The Astrophysical Journal*, 689 (2008): pp.1044.

[7] Owen Gingerich, The Origins of Kepler's Third Law', *Vistas in Astronomy* 18 (1975): pp.595–601.

[8] Jo Peter Pesic, 'Earthly Music and Cosmic Harmony: Johannes Kepler's Interest in Practical Music, Especially Orlando di Lasso', *Journal of Seventeenth Century Music,* 11 (2005): No. 1

[9] Michael Dickreiter, *Der Musiktheoretiker Johannes Kepler* (Bern and Munich: Francke Verlag, 1973).

[10] Johannes Kepler, *Harmonice mundi,* trans. E. J. Aiton, A. M. Duncan, and J. V. Field (Philadelphia, PA: The American Philosophical Society, 1997).

[11] Aiton, Duncan, and Field, *Harmonice Mundi*, p.139.

many of those observations may have come from his own acoustic experiments involving the detection of harmonies by the ear and measurements of the lengths of vibrating strings.[12] The remarkable precision of measurements of lengths of strings given by Kepler is vividly evident in a comparison with modern measurements of frequency in Table 1.[13] I have converted length to frequency using A as the common measure. We know he made acoustic experiments with strings of different lengths, and it is possible that he made some those measurements given in Table 1.[14]

	Kepler Lengths of strings	Kepler Frequency	Modern Frequency	% Difference
G	1080	**195.9**	**195.6**	-0.2
F#	1152	**184.9**	**183.3**	-0.8
F	1215	**174.6**	**173.8**	-0.4
E	1296	**164.8**	**163.0**	-1.1
Eb	1350	**155.5**	**156.4**	0.6
D	1440	**146.8**	**146.7**	-0.1
C#	1536	**138.6**	**137.5**	-0.8
C	1620	**130.8**	**130.4**	-0.3
B	1728	**123.5**	**122.2**	-1.0
Bb	1800	**116.5**	**117.3**	0.7
A	1920	**110.0**	**110.0**	0.0
G#	2048	**103.8**	**103.1**	-0.7
G	2160	**98.0**	**97.8**	-0.2

Table 1. Bass Clef: Comparison of Kepler's and Modern Values (the 3/2 interval of the perfect fifth is between D and G).

[12] 'I follow the evidence of my ears', Aiton, Duncan, and Field, *Harmonice Mundi*, p.65.
[13] Aiton, Duncan, and Field. *Harmonice Mundi*, p.196.
[14] Daniel Walker, *Studies in Musical Science in the Late Renaissance* (London: Warburg Institute, 1978), p.48.

How did he eventually come up with 3/2? This musical interval, the perfect fifth is the most harmonious of all intervals next to the octave. It is one of the easiest intervals for humans to learn to sing and for string players to identify. The upper note makes three vibrations in the same amount of time that the lower note makes two. The perfect fifth can be heard when a stringed instrument, violin, viola, or cello, is tuned. Their strings are each separated by perfect fifths as demonstrated in Table 2. When tuning their instruments, musicians are, in a sense, placing themselves into resonance with the harmonies of the astronomical universe. The perfect fifth joined with the four fingers of human left hand are responsible for the design of these stringed instruments.

The perfect fifth is also crucial in tuning the piano according to Ted Mulcahey of the College of Music of the University of Colorado: 'The perfect fifth is the most important interval to me (with the octave a close second) in aural tuning and piano tuning in general. I strive to get the cleanest sounding fifths possible without compromising too much on the octaves. [15]

Violin	**Frequency**	**Ratio of upper frequency to lower frequency**
E5	659.3	1.4984
A4	440	1.4981
D4	293.7	1.4985
G3	196	
Viola		
A4	440	1.4981
D4	293.7	14985
G3	196	1.4985
C3	130.8	
Cello		
A3	220	1.4986
D3	146.8	1.4980
G2	98	1.4985
C2	65.4	

Table 2. The 3/2 ratio of stringed instruments

[15] Personal communication Ted M Mulcahey, Head Piano Technician, University of Colorado College of Music (5/2/2022).

How many attempts he made in testing his idea, we have no way of knowing.[16] It is significant that Kepler, himself, defined two additional musical intervals: the lower imperfect fifth (40/27=1.48) and the greater imperfect fifth (243/160=1.518).[17] These two intervals lie on either side of 1.5, and he may have been using them to test how precise is the fit of 3/2 to Tycho's data (Table 3). Kepler may have been following the procedures of modern experimental scientists to check and double check their results. Carl Sagan called him the 'first astrophysicist and the last scientific astrologer'.[18] Kepler understood the tradition of the Pythagoreans, which linked heaven and earth through music, although he distrusted their mysticism insisting on the preeminence of observational data.[19] That data, of course, came from the observation of Tycho Brahe and was consistent with a law involving the perfect fifth. But he wanted a deeper understanding. He was interested in root causes and physical explanations.

Why did the planets move around the sun? He believed they were pushed by a physical force, which he apparently suspected was that of magnetism, which had recently been described by Gilbert in 1600.[20] The pushing was produced by filaments of magnetism that emanated from the rotating sun. His intuition was wrong about the motion of planets and the effects of gravity, but entirely without realizing it, he did anticipate those plumes of magnetic field and charged particles in the solar wind and those that are ejected (coronal mass ejections) by magnetic instabilities on the rotating sun that travel through the interplanetary medium.When he learned about Galileo's discovery of moons revolving around Jupiter, he was pleased to learn that Jupiter was also rotating and could thus drive its moons in a manner like the sun. Again, his physics was wrong, but he was thinking like an astrophysicist.

His eventual discovery that the ratio 3/2 was the key to understanding the behavior of the planets sent him into a "sacred frenzy" which Casper

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[16] Arthur Miller.. *Deciphering the cosmic number: the strange friendship of Wolfgang Pauli and Carl Jung*. (New York: W. W. Norton & Company, 2009). pp.79–80.

[17] Stephen Hawking (ed.), *On the Shoulders of Giants* (New York: Running Press 2004), p.22.

[18] Carl Sagan, *Cosmos* Harmony of the World, Episode 3, 12 October 1980.

[19] , Kenneth S. Guthrie and David R. Fideler, eds, *The Pythagorean Sourcebook and Library: An Anthology of Ancient Writings which Relate to Pythagoras and Pythagorean Philosophy* (Grand Rapids. MI: Phanes Press, 1987).

[20] Christopher Linton, *From Eudoxus to Einstein. A History of Mathematical Astronomy* (Cambridge: Cambridge University Press, 2004), pp.183–84.

describes as the rapture 'not the exaltation of a dreamer; it is the enthusiasm of the discoverer'.[21] Not only did he discover the mathematical law describing the behavior of the planets, but the mathematics of that law turned out to be identical to that of the harmony of the perfect fifth. This combination of mathematics, music, and the physical world will reappear throughout the succeeding four centuries of humankind's exploration of the physical world.

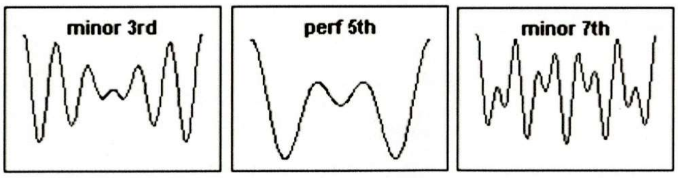

Fig. 1. Wave forms of Intervals as Observed in an Oscilloscope (modified, Music Department, SUNY ONEONTA).[22]

The perfect fifth is harmonious because of the way the waveforms of its two notes interact with each other. Next to the octave, it has the simplest wave form with the fewest peaks and valleys, resulting in a smooth sound. The more peaks and valleys in a wave form the greater the 'grinding' sound. Access to an oscilloscope would have satisfied Kepler's yearning to understand the fundamental meaning of harmony.

Substantial evidence for the role of the perfect fifth in Kepler's discovery is found in the original Latin of *Harmonice mundi:* 'Sed res est certissima exactissimaque quod proportio qua est inter binorum quorumcunque Planetarum tempora periodica, sit præcise sesquialtera proportionis mediarum distantiarum, ...'.[23] Note that he employed the word sesquialtera. Its musical counterpart in Greek is *hemiholia.*

[21] Max Casper, *Kepler*, trans. C. Doris Hellman (New York: Dover Publications, 1993), pp.267–68.
[22] Michael Lopresto, 'Using musical intervals to demonstrate superposition of waves and Fourier analysis', *Physics Education,* 48:640 (2013).
[23] Johannes Kepler, *Harmonice Mundi* [The Harmony of the World] (Linz:: Johann Planck, 1619), p.189.

Major Second	9/8	1.125	$3^2/2^3$
Minor Third	6/5	1.2	
Pythagorean Major Third	81/64	1.26	$2^5/3^3$
Perfect Fourth	4/3	1.33	$2^2/3$
Augmented Third	177147/131072	1.352	$3^{11}2^{17}$
Diminished Fifth	1034/729	1.4047	$2^{10}/3^6$
Augmented Fourth	729/512	1.4238	$3^6/2^9$
Diminished Sixth	262144/177147	1.4798	$2^{18}/3^{11}$
Lower Imperfect Fifth	40/27	1.481	$2^5/3^3$
Perfect Fifth	**3/2**	**1.5**	**3/2**
Greater Imperfect Fifth	243/160	1.518	
Minor Sixth	128/81	1.5802	$2^7/3^4$
Augmented Fifth	6561/4096	1.6018	$3^8/2^{12}$
Major Sixth	27/16	1.68	$3^3/2^4$
Minor Seventh	16/9	1.77	$2^4/3^2$

Table 3. Pythagorean and Other Intervals: ratios of upper and lower frequencies.

It is unfortunate that in their translation of *Harmonice Mundi*, E. J. Aiton, A. M. Duncan, and J. V. Field replaced sesquialtera by sesquialterate Their translation: 'But it is absolutely certain and exact that "the proportion between the periodic times of any two planets is precisely the sesquialterate (3/2) proportion of their mean distances…'.[24]

In order to describe the 3/2 ratio, Kepler had a choice between two terms: sesquialtera and sesquialterate. Both come from *sesquialter,* meaning one and a half, but they have different meanings in different contexts. Sesquialterate is an archaic mathematical term for the ratio of one to a half of one, and it satisfies the mathematical needs of the law. Sesquialtera describes the musical attributes of that ratio, namely the perfect fifth. The early Pythagoreans, such as Hippasus and Philolaus, used this term in the context of music theory to mean the perfect fifth.[25] Kepler used *hemiola* and *sesquialtera* both of which signify the ratio 3:2 in music.[26]

[24] Aiton, Duncan, and Field, *Harmonice Mundi*, p.411.
[25] Andrew Barker, *Greek Musical Writings: [vol. 2] Harmonic and Acoustic Theory* (Cambridge: Cambridge University Press, 1989), pp.31, 37–38.
[26] Henry George Liddell and Robert Scott, *A Greek-English Lexicon*, 9th edn (Oxford: Clarendon Press, 1940).

Sesquialtera contains within it the memories of the inspiration and labor that led Kepler to his discovery, whereas sesquialterate is simply a mathematical expression, devoid of the cultural context of discovery. In their justification for the substitution, Aiton, Duncan, and Ford state: 'Sesquialtera proportino and cognate terms…. have been translated by the obsolete term "sesquialterate" because to paraphrase them by translations as "the proportion of one to one and a half' would be intolerably clumsy and would make it even harder for the reader to follow Kepler's arguments'.[27] Their use of "sesquialterate" leaves the reader with the impression that the **third law** had its origin in mathematics rather than music. That does not appear to be the case. Music was most likely the preeminent inspiration for Kepler's inductive leap. In his analysis of Riedweg's book on Pythagoras George Latura also asserts that the perfect fifth was important for Kepler.[28]

Planet	**a (AU)**	**Period**	**Perfect Fifth: a^3/P^2 x 10^{-6}**	**Perfect Fourth: a^4/P^3 x 10^{-8}**	**Lower imperfect fifth a^{40}/P^{27} x 10^{-71}**
PM	.389	87.77	7.64	3.39	134
V	.724	224.7	7.52	2.42	78.6
E	1	365.25	7.50	2.05	64.6
M	1.524	686.95	7.49	1.66	52.8
J	5.2	4332.62	7.43	.899	28.0
S	9.51	10759.2	7.43	.657	18.6

Table 4. Planets, the Perfect Fifth, the Perfect Fourth, the Lower Imperfect Fifth

For Kepler harmony was a real process entwining music and mathematics that had been placed in the cosmos for us to discover. Rothman suggests that the search for harmony became more important to Kepler than making

[27] Aiton, Duncan, and Field, *Harmonice Mundi*, p.xii.

[28] George Latura, 'Kepler's Sesquialter & the Tetraktys of Pythagoras', paper presented at the American Astronomical Society Meeting January 2020 Honolulu; Christoph Riedweg, *Pythagoras, His life, teaching, and influence,* trans. Steen Rendall (Ithica, NY: Cornel University Press, 2005), p.131.

discoveries in astronomy.[29] His hope was that the demonstration of harmony in the heavens could bring about harmony on earth. Sadly, eight days after 15 May 1618, the defenestration of Prague occurred, and the disastrous Thirty Years War commenced. In that same month, the first hearing of witnesses for trial of his mother as a witch began.[30]

One might think that Kepler in his subsequent books, especially the *Rudolphine Tables,* would have used his third law to compute the mean solar distances of the planets from their sidereal periods, which would have produced their accurately. Linton suggest he may have been concerned about using a law that was inspired by his speculations about harmony rather than based primarily upon observation.[31] The third law initially seemed to be of little use in practical astronomy and did not attract much attention in the mid seventeenth century, until Thomas Streete, one of the leading astronomers working in England at the time, drew attention to it. In his *Astronomia Carolina* Streete used the law to calculate the mean solar distances of the planets from their sidereal periods, leading to improved accuracy for Mercury, Venus, and Mars.[32] It was from Street's work that Isaac Newton, while an undergraduate student at Trinity College, became aware of Kepler's third law, the details of which he copied (page r29) into his Trinity College Notebook (1661–1665). The notebook contains, among other items, notes on books he was recommended to read for his studies.[33] Within in a remarkably short period of time Newton was able to use Kepler's ideas to develop the Law of Gravity.

Power Laws

In his third law, Kepler discovered the world's first data-driven power law. A power law is a relationship in which one quantity varies as a power of

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[29] Aviva Rothman, 'Johannes Kepler and the Pursuit of Harmony', *Physics Today,* 73.1, (2020): pp.36-42.

[30] Max Casper, *Kepler,* transl. C. Doris Hellman (New York: Dover Publications, 1993), p.286.

[31] Linton, *From Eudoxus to Einstein. A History of Mathematical Astronomy.*

[32] Thomas Streete, 1661, *Astronomia Carolina: a new theorie of the cœlestial motions : composed according to the best observations and most rational grounds of art, yet far more easie, expedite and perspicuous than any before extant : with exact and most easie tables thereunto, and precepts for the calculation of eclipses, &c.* (Londin, Lodowick Lloyd)

[33] Cambridge Digital Library, cudl.lib.ca.ac.uk/collections/newton; MS Add. 3996.

another. Elsewhere on the Earth we find power laws in distributions of avalanches, the energy released by solar flares, earthquakes, the coastline of Norway, forest fires, the sizes of meteorite craters, micrometeorites, and the diameters of trees in mature forests.[34 35]

He discovered the scale invariance of power laws when he found that the third law worked equally well for both the solar system and the moons of Jupiter, even though the two systems differ by 10,000 in scale. (Fig. 2) It is their property of *scale invariance* that makes power laws so invaluable in astronomy, ranging from moons in our solar system to supermassive black holes orbiting each other in distant galaxies.

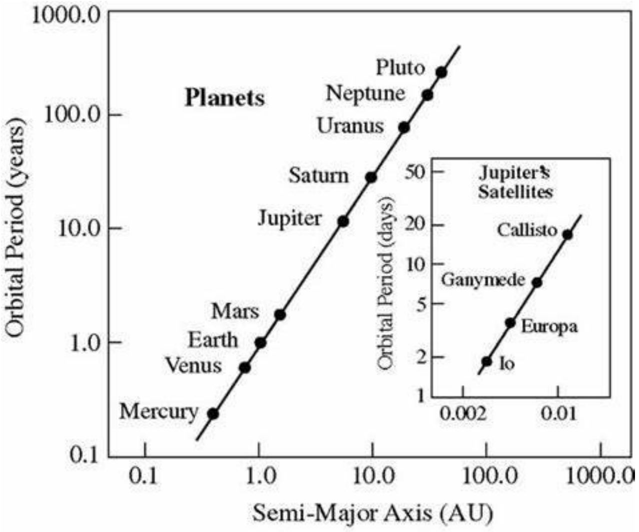

Fig. 2. The Planets and the Moons of Jupiter (Tufts University).

[34] Donald Turcotte, *Fractals and Chaos in Geology and Geophysics*, (Cambridge: Cambridge University Press, 1997); James Sethna, 'Power Laws in Physics', *Nature Reviews* 4 (2022): pp.501–3.
[35] Per Bak, *How Nature Works: the science of self-organized criticality* (New York: Springer-Verlag, 1996).

The Harmony of Jupiter's Moons

As of 2023, there are 92 identified moons of Jupiter.[36] The most massive of the moons are the four Galilean moons: Io, Europa, Ganymede, and Callisto, which were discovered in 1610 by Galileo Galilei and Simon Marius. These were the first objects found to orbit a body that was neither Earth nor the Sun, and, most wonderfully, they also obey the third law (Fig. 2).

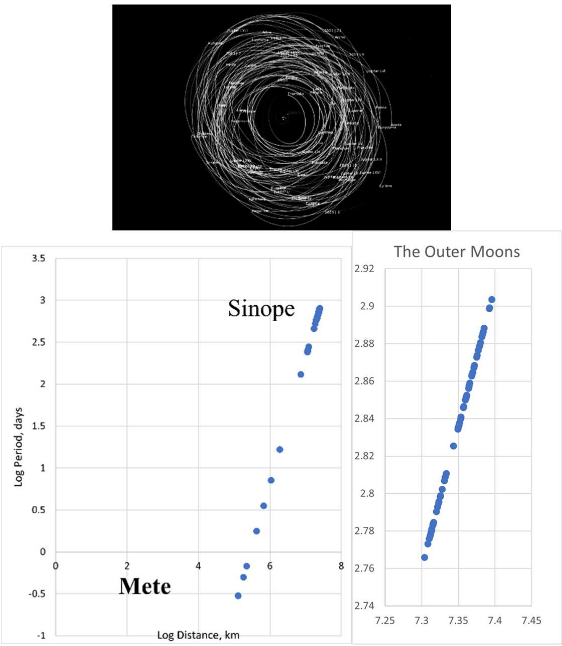

Fig. 3. Moons of Jupiter.[37] a. The view from the top; b. The viewpoint of the **third law**.

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[36] Scott S. Sheppard, *"Moons of Jupiter". Earth & Planets Laboratory*. Carnegie Institution for Science. Retrieved 7 January 2023.

[37]

Many of the moons appear to have been captured fragments of asteroids, comets and other random objects that wandered too close to Jupiter, yet they all were organized according to Kepler's third law. If one plots the orbits of the moons based simply upon their locations in space the picture looks like a messy collection of spaghetti (Fig. 3a). But when one puts on Keplerian glasses and organizes the moons according to the third law, the change is from one of chaos to that of order and harmony (Fig. 3b).

Newton and the Law of Gravity

For Newton to develop his law of universal gravitation he needed to connect mathematically the period of a planet with its distance from the sun.[38] Newton knew that a planet circling the sun was accelerating at the rate v^2/r. Once, he had determined that the speed of the planet was $2\pi r$ divided by the period, he established that the acceleration of the planet the sun was $4\pi^2 r/P^2$. But he was stuck. Is there a relation between distance and period? Ah, yes, he remembered; that relation was contained in Kepler's **third law**: $r^3 = P^2$. Replace P^2 by r^3 and add F=ma, one arrives at F= $m4\pi^2/r^2$, which is the inverse-square Law of Gravity, voilà! Newton's derivation of the Law of Gravity rested 'squarely upon Kepler's third law'.[39] [40]

In Newton's words:

> *In the beginning of the year 1665, I found the method of approximating series and the Rule for reducing any dignity of any Binomial into such a series. The same year in … November had the direct method of Fluxions, and in January had the Theory of Colours, and in May following I had entrance into the inverse method of Fluxions. And the same year I began to think of the orb of the Moon … from Kepler's Rule of the periodical times of the Planets* ***being in sesquialterate proportion of their distances from the center of their Orbs,*** *I deduced the forces which keep the Planets in their orbs must be reciprocally as the squares of their distances from the centres about which they revolve … All this was in the two*

[38] Richard Westfall, *Never at Rest* (Cambridge: Cambridge University Press, 1980), p.152; Kenneth Ford, *Basic Physics* (Waltham, MA: Blaisdell, 1968), pp.352–61.

[39] Richard Westfall, *Never at Rest: A Biography of Isaac Newton,* pp.352–61.

[40] In today's world for some researchers, the dependence by Newton upon Kepler might have resulted in the Law of Gravity being identified as the Newton-Kepler Law of Gravity.

> *plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded Mathematics and Philosophy more than any time since.*[41]

It was Kepler's third law, not the apple falling off a tree, that seemingly led Newton to his Law of Gravity. In his biography of Newton, Richard Westfall comments that this story of Newton watching an apple fall 'vulgarizes universal gravity by treating it as a bright idea. A bright idea cannot shape a tradition'.[42] During the time leading up to his deduction, Newton had been 'baffled for the moment by overwhelming complexities'.[43] For example, he considered the idea that fluid vortices may have produced the behavior of the planets. Newton argued that periods of revolution in a vortex vary as the square of the radius, whereas Kepler's third law based on observational data, required the three-halves power. Furthermore, he argued that the variations in velocity in a vortex cannot correspond to the second law: 'so that the hypothesis of vortices is utterly irreconcilable with astronomical phenomena'.[44]Again, Kepler led Newton in the right direction.

Kepler also came to the rescue of Newton in connection with his unfortunate conflict with Robert Hook concerning his charge of plagiarism against Newton. Hook had claimed priority in the derivation of the inverse-square law based upon, in the words of Westfall, "a bastard demonstration resting up a deep confusion about dynamics and accelerated motion."[45] Newton was not inventing gravity inspired by a 'eureka' moment but had assiduously analyzed the observational data of Tycho and Kepler. Newtonian gravity was data-driven. Hook's claim could have no standing in a court of inductive science.

The Mass of Stars

At least half of the stars in the sky are contained in binary and multiple star systems. Therein lies another story of the power of Kepler's third law in its ability to measure the mass of stars. If all of the stars in our sky were single, we would be helpless to understand the nature of these self-luminous objects. Newton was able to show that Kepler's equation could be arranged

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[41] Westfall, *Never at Res*t, p.143.
[42] Westfall, *Never at Rest*, p.155
[43] Westfall, *Never at Rest*, p.155
[44] Newton, *Principia*, pp.395–96.
[45] Westfall, *Never at Rest*, pp 386–87.

such that the mass around which planets move is given by M $= a^3/ P^2$ in solar system units where a is given in astronomical units and P is in years. In a visual binary system, in which both stars are visible, and if their distance is measurable, the mass of individual stars can be determined.[46] This ability leads to one most important sets of data in astronomy, the mass-luminosity relation. Much of what we now know about the lifetime, structures, and temperatures of stars has been built upon the mass luminosity relationship, which is another power law. (Fig. 4)

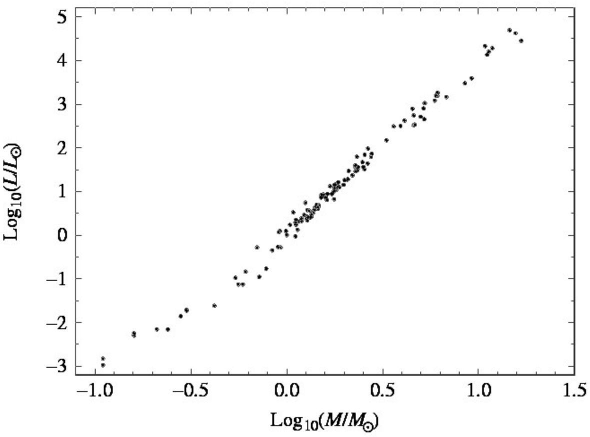


Fig. 4. Mass Luminosity Relation.

Dark Matter in Galaxies

The existence of dark matter was discovered by Vera Rubin[47] and her colleagues, by measuring the rotation of galaxies. They found that galaxies were rotating too fast to be stable, based upon mass determined by visible stars. Stars should fly away like sparks on a spinning grinding wheel. The

[46] Daniel Popper, 'Stellar Masses', *Annual Review of Astronomy and Astrophysics*, 18 (1980): p.115.

[47] Vera Rubin and W. Kent Ford, Jr., 'Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions', *The Astrophysical Journal*, 159 (1970): pp.379ff. Vera Rubin,. 'A Century of Galaxy Spectroscopy', *The Astrophysical Journal*. 451 (1995): pp. 419ff.

fast rotation of galaxies is best explained by the existence of a substantial amount of dark matter, which holds everything together by the gravity it produces.

Dark matter is also found in dwarf galaxies. In our Local Group of galaxies, there are some dwarf galaxies in which dark matter makes up 90% of the matter.[48] Our Local Group is dominated by two large galaxies, our Milky Way and the somewhat larger Andromeda Galaxy. While the rest of the universe is expanding, we and Andromeda are glued together firmly by dark matter. The amount of intergalactic matter in our local group is estimated by Kepler's third law to be 4 trillion solar masses, of which most is dark matter.[49]

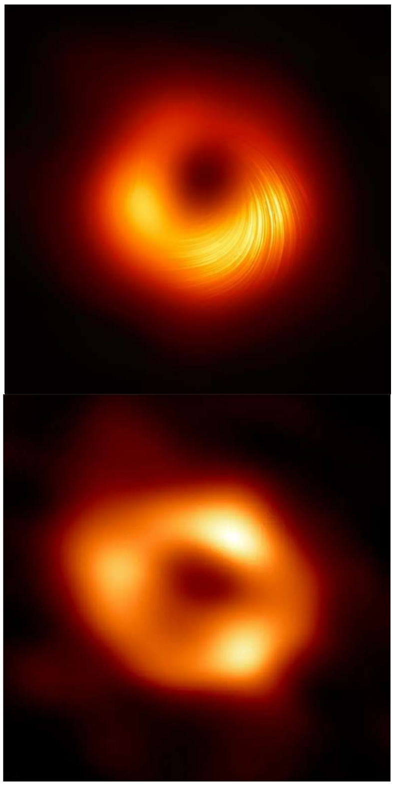
The Mass of Giant Black Holes

Another fantastic triumph of his third law, lying far beyond the imagination and dreams of Kepler who had himself authored a book of science fiction[50], is found in the centers of the elliptical galaxy known at M87 and our Milky Way galaxy. The mass of the huge black hole in the center of M87 was first estimated by measurement of the revolution of stars around it, following Keplerian orbits, giving a value of over 6 billion solar masses[51].

	Sagittarius A*	M87	M87/SagA*
Mass (solar mass) and therefor size	$4.15x10^6$	$6.4x10^9$	$1.5x10^3$
Distance (light years)	$26x10^3$	$53.5x10^6$	$2.05x10^3$
Size in the sky (microarcseconds)	51.8	42	

Table 3. Comparison of M87 and SagA*

[48] Bradley Carroll and Dale Ostlie, *An Introduction to Astrophysics* (Cambridge: Cambridge University Press 2017), pp.1060–61.
[49] Bradley Carroll and Dale Ostlie, *An Introduction to Astrophysics,* p.1061.
[50] Isaac Asimov and Carl Sagan, suggest that the genre of science fiction began with Kepler's novel called *Somnium* ('The Dream'). John Lear, *Kepler's Dream*, with the full text and notes of "Somnium, Sive Astronomia Lunaris, Joannis Kepleri," translated by Patricia Frueh Kirkwood (Berkeley, CA: University of California Press, 1965).
[51] Karl Gebhardt, Joshua Adams, Douglas Richstone, Tod R. Lauer, S. M. Faber, Kayhan Gültekin, Jeremy Murphy and Scott Tremaine, 'The Black Hole Mass in M87 from Gemini/NIFS Adaptive Optics Observations', *The Astrophysical Journal,* 729 (2011): p.119.



a b

Fig 5. a. The Black Hole in the Center of M87 in polarized light; b. The black hole in the center of the Milky Way galaxy (The Event Horizon Telescope Collaboration).

The motion of stars near the center of our own Milky Way provides evidence that these stars are orbiting a supermassive black hole, now identified as Sagittarius A*. Since 1995, the motions of 90 stars orbiting an invisible object have been. (Fig. 6) They follow hybrid Keplerian-like orbits, departing from pure Keplerian ellipses when, close to the back hole, where General Relativity takes over from Newton and Kepler.[52] The star S2 displays Schwarzschild precession, similar to, but some 100 times faster than the precession of Mercury, once incorrectly attributed to the non-existent planet Vulcan.[53] It is extraordinary that Kepler's analysis of planets orbiting our sun some 4 centuries ago should apply to stars orbiting a black hole in the center of our galaxy. The scale independence of the third law applies to these supermassive black holes. Although they differ in mass by a thousand, they look alike (Table 3 and Fig. 5).

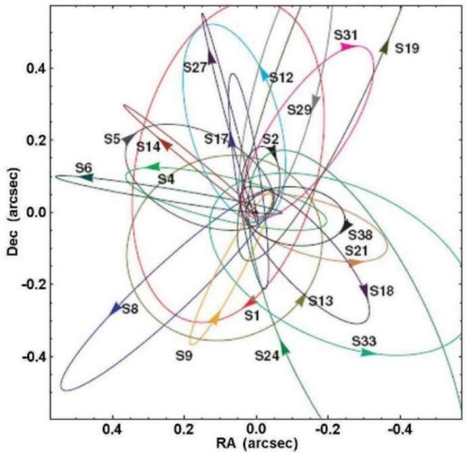


Fig. 6. Hybrid Keplerian Orbits of Stars Around the Black Hole in the Center of our Galaxy. (Gillessen et al., 2010)[54]

Recent evidence for supermassive blackhole binaries has added an even more impressive example of Kepler's power law. Some galaxies contain

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[52] R. Genzel et al. 'Detection of the Schwarzschild precession in the orbit of the star S2 near the Galactic center's massive black hole' *Astronomy and Astrophysics*, 636, L5 (2020).
[53] Thomas Levenson, *The Hunt for Vulcan* (New York: Random House, 2016).
[54] S. Gillessen et *al*., 2009, 'Monitoring Stellar Orbits Around The Massive Black Hole In The Galactic Center'; Ghez et al., 'Measuring Distance and Properties of the Milky Way's Central Supermassive Black Hole with Stellar Orbits', pp.1044.

supermassive black holes in their center, containing billions of solar masses. When they collide, these supermassive black holes may join in becoming a huge binary system, circling each other in Keplerian orbits that are light years across. As they circle, they generate low frequency gravity waves which apparently have been detected by their effects on pulses of millisecond pulsars that rotate thousands of times per second.[55] Far from each other they obey the third law in their orbiting, but as they get closer to each other they will follow the General Theory of Relativity (as does Mercury at perihelion), probably ending their separate ways in an immense burst of light and gravity waves.

Concluding Remarks

A major conclusion of this paper is the centrality of music, in the form of the perfect fifth, in the discovery of one of the great power laws of astronomy and astrophysics. Using the Latin term *susquialtera* Kepler reveals the importance of the perfect fifth in his discovery. What a small, elegant, and powerful expression is that third law, which, after being modified by Newton, has given astronomers quantitative access to an extraordinary variety of phenomena in the universe.

The ideas of Kepler have sadly not brought the harmony to our world that he so earnestly sought. But his spirit continues to express harmony in the heavens. In many ways, the most remarkable aspect of the third law is that it was not primarily intended by Kepler to reveal new knowledge about the universe but rather to encourage humans to live peacefully and harmoniously with each other. That search for harmony in the universe has continued over the past four centuries, reaching its climax in Einstein's General Theory. Speaking at the German Physical Society in 1918, Albert Einstein described the driving force behind his work as 'the longing to behold *pre-existing harmony*'.[56]

[55] Gabraiella Agazie, et al., 'The NANOGrav 15 yr Data Set: Constraints on Supermassive Black Hole Binaries from the Gravitational Wave Background, as Xiv 2306.16220', *The Astrophysical Journal Letters,* V 952, (2) L37 (2023).
[56] Abraham Pais, *Subtle is the Lord* (Oxford: Oxford University Press, 1982), pp.26–27.